Hall coefficient of equilibrium supercurrents flowing inside superconductors

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We study augmented quasiclassical equations of superconductivity with the Lorentz force, which is missing from the standard Ginzburg-Landau and Eilenberger equations. It is shown that the magnetic Lorentz force on equilibrium supercurrents induces finite charge distribution and the resulting electric field to balance the Lorentz force. An analytic expression is obtained for the corresponding Hall coefficient of clean type-II superconductors with simultaneously incorporating the Fermi-surface and gap anisotropies. It has the same sign and magnitude at zero temperature as the normal state for an arbitrary pairing, having no temperature dependence specifically for the *s*-wave pairing. The gap anisotropy may bring a considerable temperature dependence in the Hall coefficient and can lead to its sign change as a function of temperature, as exemplified for a model *d*-wave pairing with a two-dimensional Fermi surface. The sign change may be observed in some high-T_c superconductors.

I. INTRODUCTION

Einstein[1] pointed out in 1905 that the Lorentz force in electromagnetic fields can be deduced naturally from the self-evident force on a charge at rest in an electric field with his theory of special relativity. He has thereby provided a firm logical ground on the magnetic part of causing a deflection. However, this force is absent in the modern theoretical accomplishments of superconductivity, i.e., the standard Ginzburg-Landau equations[2,3] and the quasiclassical Eilenberger equations[4–7] derived microscopically from the Gor'kov equations.[3,5–7] Thus, our understanding on the magnetic Lorentz force in superconductors has remained at a somewhat phenomenological level. We here wish to make an improvement on this fundamental issue, focusing our attention on equilibrium cases.

London[8] included the Lorentz force as a necessary ingredient in his phenomenological equations of superconductivity. They predict that an equilibrium supercurrent $\mathbf{j}_s \equiv en_s\mathbf{v}_s$ in a magnetic field $\mathbf{B}$ accompanies an electric field,

$$\mathbf{E} = \frac{m}{2e}\boldsymbol{\nabla} v_s^2 = \frac{1}{n_s ec}\mathbf{B} \times \mathbf{j}_s, \tag{1}$$

with m as the electron mass, e (<0) as the charge, $\mathbf{v}_s$ as the superfluid velocity, n_s as the superfluid density, and c as the light velocity. The second equality results from the London equation $\boldsymbol{\nabla} \times \mathbf{v}_s = -(e/mc)\mathbf{B}$ with the condition $(\mathbf{v}_s \cdot \boldsymbol{\nabla})\mathbf{v}_s = \mathbf{0}$. The expression implies that one could estimate the superfluid density n_s through the Hall coefficient $(n_s ec)^{-1}$, which would diverge toward the transition temperature T_c. On the other hand, van Vijfeijken and Staas[9] presented phenomenological two-fluid equations with the Lorentz force, which modify Eq. (1) into

$$\mathbf{E} = \frac{n_s}{n}\frac{m}{2e}\boldsymbol{\nabla} v_s^2 = \frac{1}{nec}\mathbf{B} \times \mathbf{j}_s, \tag{2}$$

where n is the electron density. Thus, the Hall coefficient is predicted to stay constant up to T_c contrary to the London theory. These considerations with the free-electron dispersion were extended by Adkins and Waldram[10] to incorporate the electronic band structure from a somewhat different context of the Bernoulli potential, with no explicit connection to the Lorentz force. Specifically, they considered how a uniform supercurrent at $T=0$ modifies the Cooper pairing of nonspherical Fermi surfaces to present an expression of the Hall coefficient, which can take either sign just as the one of the normal state. Hong[11] and Omel'yanchuk and Beloborod'ko[12] later performed microscopic calculations of the equilibrium electric field due to an almost uniform supercurrent, also with no direct relevance to the Lorentz force. Using the Gor'kov equations with the free-electron density of states, they obtained an expression in favor of Eq. (2) together with an additional term. However, all the finite-temperature effects in their derivations originate from the subtle energy dependence of the free-electron density of states, so that they might be deduced to vanish for a constant density of states near the Fermi level. It should be noted finally that no investigations seem to have been carried out for the cases of anisotropic pairings.

Pioneered by Onnes and Hof in 1914, efforts have also been made to detect an equilibrium/quasiequilibrium Hall voltage of superconductors.[13–18] One can show with Eq. (1) and (2), the Maxwell equation $\boldsymbol{\nabla} \times \mathbf{B} = (4\pi/c)\mathbf{j}_s$, and the condition $(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B} = \mathbf{0}$ that the Hall voltage V_H in the Meissner state between the surface of the sample and its interior is given by

$$V_H = \frac{cR_H}{8\pi}B_0^2, \tag{3}$$

with R_H the Hall coefficient and B_0 the external field. It could be detected with a spheroid sample in a longitudinal magnetic field by measuring the potential difference between a point on the equator and a pole;[19] see Refs. 14 and 17 for the experimental setup. However, early experiments[13–16] observed null Hall voltage contrary to the theoretical predictions. Hunt[20] and Nozières and Vinen[21] later pointed out independently that voltmeters used in those experiments, which require direct contacts to the sample, are not appropriate to detect the electrostatic potential. Indeed, voltmeters

can only pick out the chemical-potential difference, but the chemical potential is constant in equilibrium throughout the sample. The difficulty was circumvented successively by applying capacitive couplings to the specimen.[17,18] Bok and Klein[17] performed a low-temperature measurement of the Hall voltage with Pb as well as Nb and PbIn below H_{c1} to obtain a good agreement of their results with Eq. (1). Morris and Brown[18] carried out a detailed experiment on Pb up to T_c to report that their data point to Eq. (2) rather than Eq. (1). However, detailed experiments over a wide range of materials still seem required to establish the sign and the magnitude of the superconducting Hall coefficient in connection with the normal-state one. Especially, no experiments seem to have been carried out on materials with anisotropic energy gaps such as high-T_c superconductors where new physics may be expected.

It was shown recently that the Lorentz force can be incorporated appropriately into the quasiclassical equations of superconductivity starting from the Gor'kov equations in the Keldysh formalism.[22] The key procedures were: (i) an extension of the gauge-invariant Wigner transformation introduced by Stratonovich[23] and Fujita[24] for the normal state to the Nambu Green's function; and (ii) a derivation of the corresponding Groenewold-Moyal product[25,26] for performing the gradient expansion. They have successfully removed the imperfect gauge invariance in a couple of preceding treatments.[7,27,28] The resulting equations can describe both the equilibrium and dynamical behaviors of superconductors with the Lorentz force such that the normal-state Boltzmann equation is included appropriately as a limit. Using them, we here develop a microscopic theory of the Lorentz force on equilibrium supercurrents with the Fermi-surface and gap anisotropies. We will thereby clarify: (i) the validity or applicability of the phenomenological results of Eqs. (1) and (2); and (ii) how the gap anisotropy affects them. This step will also be necessary before elucidating dynamics of superconductors microscopically where there still remain many unresolved issues directly connected with the Lorentz force.[21,29–33]

This paper is organized as follows. Section II presents the augmented quasiclassical equations of superconductivity with the Lorentz force. Section III derives the expression of the Hall coefficient of equilibrium supercurrents. Section IV presents its temperature dependence for both the *s*-wave and *d*-wave pairings on a model two-dimensional Fermi surface. Section V provides a brief summary.

II. AUGMENTED EILENBERGER EQUATIONS

For simplicity, we first restrict ourselves to clean weak-coupling *s*-wave superconductors in equilibrium. The corresponding quasiclassical equations of superconductivity, augmented so as to include the Lorentz force, are given by[22]

$$[\varepsilon\hat{\tau}_3 - \hat{\Delta}, \hat{g}^{R,K}] + i\hbar\mathbf{v}_F \cdot \boldsymbol{\partial}\hat{g}^{R,K}$$
$$+ \frac{i\hbar}{2}\left[e\mathbf{v}_F \cdot \mathbf{E}\frac{\partial}{\partial\varepsilon} + \frac{e}{c}(\mathbf{v}_F \times \mathbf{B}) \cdot \frac{\partial}{\partial\mathbf{p}_F}\right]\{\hat{\tau}_3, \hat{g}^{R,K}\} = \hat{0}.$$

(4)

Here $\hat{g}^{R,K} = \hat{g}^{R,K}(\varepsilon, \mathbf{p}_F, \mathbf{r})$ are the 2×2 retarded and Keldysh

Green's functions, respectively, ε denotes the excitation energy, $\hat{\tau}_3$ the third Pauli matrix, $\hat{\Delta} = \hat{\Delta}(\mathbf{r})$ the gap matrix, $\mathbf{v}_F$ the Fermi velocity, $\mathbf{p}_F$ the Fermi momentum, $[\hat{P}, \hat{Q}] \equiv \hat{P}\hat{Q} - \hat{Q}\hat{P}$, and $\{\hat{P}, \hat{Q}\} \equiv \hat{P}\hat{Q} + \hat{Q}\hat{P}$. The quantity $\boldsymbol{\partial}$ denotes ∇, $\nabla - i\frac{2e}{\hbar c}\mathbf{A}(\mathbf{r})$, or $\nabla + i\frac{2e}{\hbar c}\mathbf{A}(\mathbf{r})$ when operating on the diagonal, (1,2), or (2,1) element of $\hat{g}^{R,K}$, respectively, with $\mathbf{A}(\mathbf{r})$ the vector potential. The advanced function $\hat{g}^A$ is obtained from the retarded one by $\hat{g}^A = -(\hat{\tau}_3\hat{g}^R\hat{\tau}_3)^\dagger$.

The term with $\mathbf{E}$ and $\mathbf{B}$ in Eq. (4) represents the Lorentz force which is missing from the Eilenberger equations.[4–7] It is also absent in the standard Ginzburg-Landau equations[2,3] obtained from the Eilenberger equations as a limit. Its relevance may be realized by taking the normal-state limit of $\hat{\Delta} \rightarrow \hat{0}$ and $\hat{g}^R = -\hat{g}^A = \hat{\tau}_3$; then the (1,1) element of Eq. (4) for $\hat{g}^K$ reduces to the quasiclassical Boltzmann equation in static electromagnetic fields without the collision integral and time dependence. Thus, the term is indispensable for describing dynamical behaviors of superconductors, and as seen below, will also produce observable effects even in equilibrium.

The gap matrix in Eq. (4) can be written as

$$\hat{\Delta} = \begin{bmatrix} 0 & -\Delta \\ \Delta^* & 0 \end{bmatrix}.$$

(5)

Also considering the symmetry of Eqs. (72)–(75) in Ref. 22, we can express $\hat{g}^{R,K}$ conveniently as

$$\hat{g}^R = \begin{bmatrix} g^R & -if^R \\ i\bar{f}^R & -\bar{g}^R \end{bmatrix}, \quad \hat{g}^K = \begin{bmatrix} g^K & -if^K \\ -i\bar{f}^K & \bar{g}^K \end{bmatrix},$$

(6)

where the barred functions are defined generally by

$$\bar{g}^R(\varepsilon, \mathbf{p}_F, \mathbf{r}) \equiv [g^R(-\varepsilon, -\mathbf{p}_F, \mathbf{r})]^*.$$

(7)

The elements of $\hat{g}^K$ further obey $g^{K*} = g^K$ and $\bar{f}^{K*} = f^K$.

Equation (4) is supplemented by self-consistency equations for Δ, $\mathbf{B}$, and $\mathbf{E}$ to form a closed set of equations. They are given explicitly by[4–7]

$$\Delta \ln\frac{T}{T_c} = \frac{1}{4i}\int_{-\infty}^{\infty}\left[\langle f^K\rangle - \frac{2i\Delta}{\varepsilon}\tanh\frac{\varepsilon}{2k_BT}\right]d\varepsilon,$$

(8)

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j}_s, \quad \mathbf{j}_s \equiv -\frac{eN(0)}{2}\int_{-\infty}^{\infty}\langle\mathbf{v}_F g^K\rangle d\varepsilon,$$

(9)

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \rho \equiv -\frac{eN(0)}{2}\int_{-\infty}^{\infty}\langle g^K\rangle d\varepsilon,$$

(10)

where $\langle\cdots\rangle$ denotes the Fermi-surface average with $\langle 1\rangle = 1$, k_B the Boltzmann constant, and $N(0)$ the normal-state density of states per spin and unit volume at the Fermi level. Equations (9) and (10) are just the Maxwell equations to determine the static electromagnetic fields.

The gap anisotropy can be incorporated easily into the above formalism by $\Delta(\mathbf{r}) \rightarrow \Delta(\mathbf{r})\phi(\mathbf{p}_F)$ and $\langle f^K\rangle \rightarrow \langle f^K\phi^*\rangle$ in Eqs. (5) and (8), respectively, where ϕ is the basis function on the Fermi surface with $\langle|\phi|^2\rangle = 1$. Other possible extensions will be mentioned below near the end of Sec. III with respect to the Hall coefficient.

III. ELECTRIC FIELD DUE TO MAGNETIC LORENTZ FORCE

We embark on solving Eq. (4) for the s-wave pairing by estimating the order of magnitude of the Lorentz force. To this end, let us introduce the units where the energy is measured by the energy gap Δ_0 at $T=0$ in zero fields, the length by $\xi_0 \equiv \hbar \langle v_F \rangle / \Delta_0$, the magnetic field by $B_0 \equiv \hbar c / 2|e|\xi_0^2$, and the electric field by $E_0 \equiv \Delta_0 / |e|\xi_0$. Dividing Eq. (4) by Δ_0, one may realize immediately that the magnetic Lorentz force in Eq. (4) is an order of magnitude smaller in terms of $\delta \equiv \hbar/\langle p_F \rangle \xi_0 \ll 1$. Since $\mathbf{E}$ is induced solely by the magnetic Lorentz force, as seen below, the term with $\mathbf{E}$ is also of the order of δ. It hence follows that we can carry out a perturbation expansion of Eq. (4) with respect to the Lorentz force by expanding

$$\hat{g}^{R,K} = \hat{g}_0^{R,K} + \hat{g}_1^{R,K} + \cdots . \tag{11}$$

It is performed below up to the first order in δ to an excellent approximation.

We first neglect the Lorentz force in Eq. (4) to obtain the equations of $O(\delta^0)$. They are just the standard Eilenberger equations where the solutions $\hat{g}_0^{R,K}$ satisfy $\hat{g}_0^R \hat{g}_0^R = \hat{1}$, $\bar{g}_0^R = g_0^R$, and $\hat{g}_0^K = (\hat{g}_0^R - \hat{g}_0^A)\tanh(\varepsilon/2k_B T)$.[4–7] The (1, 2) element of the equation for $\hat{g}^R$ reads

$$-i\varepsilon f_0^R + \frac{1}{2}\hbar \mathbf{v}_F \cdot \boldsymbol{\partial} f_0^R = \Delta g_0^R, \tag{12}$$

with $g_0^R = (1 - f_0^R \bar{f}_0^R)^{1/2}$, which determines the whole solution. Equation (12) with Eqs. (8) and (9) has been solved extensively to clarify vortex structures of s- and d-wave superconductors in equilibrium.[34–37]

We next consider terms of $O(\delta)$ in Eq. (4). The corresponding (1,1) and (1,2) elements of the equation for $\hat{g}_1^K$ read

$$\mathbf{v}_F \cdot \boldsymbol{\nabla} g_1^K - \frac{\Delta^* f_1^K + \Delta \bar{f}_1^K}{\hbar} = -e\mathbf{v}_F \cdot \mathbf{E}\frac{\partial g_0^K}{\partial \varepsilon} - \frac{e}{c}(\mathbf{v}_F \times \mathbf{B}) \cdot \frac{\partial g_0^K}{\partial \mathbf{p}_F}, \tag{13a}$$

$$-i\varepsilon f_1^K + \frac{1}{2}\hbar \mathbf{v}_F \cdot \boldsymbol{\partial} f_1^K = \Delta \frac{g_1^K - \bar{g}_1^K}{2}. \tag{13b}$$

The (2,2) and (2,1) elements are obtained from above by setting $(\varepsilon, \mathbf{p}_F) \rightarrow (-\varepsilon, -\mathbf{p}_F)$, taking the complex conjugate, and keeping Eq. (7) and $\bar{g}_0^K = -g_0^K$ in mind. The four equations determine g_1^K, $\bar{g}_1^K$, f_1^K, and $\bar{f}_1^K$. Writing them in terms of $g_1^K + \bar{g}_1^K$ and $g_1^K - \bar{g}_1^K$, we are led to linear closed equations for $g_1^K - \bar{g}_1^K$, f_1^K, and $\bar{f}_1^K$ without the external source. We hence conclude $f_1^K = 0$ and $\bar{g}_1^K = g_1^K$. Substitution of this result into the equation for $g_1^K + \bar{g}_1^K$ yields

$$\mathbf{v}_F \cdot \boldsymbol{\nabla} g_1^K = -e\mathbf{v}_F \cdot \mathbf{E}\frac{\partial g_0^K}{\partial \varepsilon} - \frac{e}{c}(\mathbf{v}_F \times \mathbf{B}) \cdot \frac{\partial g_0^K}{\partial \mathbf{p}_F},$$

which is clearly satisfied by the solution of

$$\boldsymbol{\nabla} g_1^K = -e\mathbf{E}\frac{\partial g_0^K}{\partial \varepsilon} - \frac{e}{c}\mathbf{B} \times \frac{\partial g_0^K}{\partial \mathbf{p}_F}. \tag{14}$$

We will use this latter equation below.

The same consideration for the equation of $\hat{g}^R$ leads to the conclusion that $f_1^R = 0$, $\bar{g}_1^R = -g_1^R$, and g_1^R is to be determined by Eq. (14) with the replacement $g_{0,1}^K \rightarrow g_{0,1}^R$. However, the solution will not be necessary below in the present clean limit.

To obtain a closed equation for $\mathbf{E}$, let us operate $\boldsymbol{\nabla}$ on Eq. (10). We then approximate $g^K \approx g_0^K + g_1^K$, substitute Eq. (14), and use $\int_{-\infty}^{\infty} \langle g_0^K \rangle d\varepsilon = 0$ resulting from $\bar{g}_0^{K*} = -g_0^K$ as well as $g_0^K \rightarrow \pm 2$ for $\varepsilon \rightarrow \pm \infty$. We thereby obtain

$$-\lambda_{\mathrm{TF}}^2 \nabla^2 \mathbf{E} + \mathbf{E} = -\frac{\mathbf{B}}{4c} \times \int_{-\infty}^{\infty} \left\langle \frac{\partial g_0^K}{\partial \mathbf{p}_F} \right\rangle d\varepsilon, \tag{15}$$

where $\lambda_{\mathrm{TF}} \equiv [8\pi e^2 N(0)]^{-1/2}$ is the Thomas-Fermi screening length.[38]

Equation (15) is one of the main results of the present paper. It enables us to calculate the induced electric field of clean superconductors in equilibrium with respect to the solution g_0^K of the standard Eilenberger equations, i.e., Eqs. (12), (8), and (9). Although derived above for the s-wave case, Eq. (15) is also valid in the presence of gap anisotropy, as seen below. It implies that the electronic screening is the same in a superconductor as its normal state. Since the source term on the right-hand side varies over the coherence length or the magnetic penetration depth which is much larger than λ_{TF}, we may generally neglect the first term on the left-hand side of Eq. (15) to an excellent approximation.

Equation (15) can be simplified further for the spherical Fermi surface with the slow-variation approximation. Let us solve Eq. (12) perturbatively up to the first order in terms of the gradient operator. Putting the result into $g_0^R = (1 - f_0^R \bar{f}_0^R)^{1/2}$, we obtain

$$g_0^R = \frac{-i\varepsilon}{W} + \frac{\Delta^* \mathbf{v}_F \cdot \boldsymbol{\partial}\Delta - \Delta \mathbf{v}_F \cdot \boldsymbol{\partial}\Delta^*}{4W^3}, \tag{16}$$

where $W \equiv \sqrt{(-i\varepsilon)^2 + |\Delta|^2}$, and an infinitesimal positive imaginary part is implied in ε. We next substitute Eq. (16) into $g_0^K = (g_0^R - g_0^A)\tanh(\varepsilon/2k_B T)$ and use it in Eqs. (9) and (15). We then find

$$\int_{-\infty}^{\infty} \left\langle \frac{\partial g_0^K}{\partial \mathbf{p}_F} \right\rangle d\varepsilon = -\frac{6\mathbf{j}_s}{mN(0)v_F^2 e} = -\frac{3c\,\boldsymbol{\nabla} \times \mathbf{B}}{2\pi mN(0)v_F^2 e}.$$

We further put this expression into Eq. (15) together with $mN(0)v_F^2 = (3/2)n$ for the free-electron model. Also neglecting the first term on the left-hand side, we obtain

$$\mathbf{E} = \frac{1}{nec}\mathbf{B} \times \mathbf{j}_s.$$

Thus, Eq. (2) by van Vijfeijken and Staas is reproduced, i.e., the superconducting Hall coefficient is predicted to stay constant up to T_c for the s-wave pairing on the spherical Fermi surface, having the same sign and magnitude as that of the normal state.

Besides the Fermi-surface anisotropy, the gap anisotropy can be incorporated easily into the above consideration by $\Delta(\mathbf{r}) \to \Delta(\mathbf{r})\phi(\mathbf{p}_F)$ and $\langle f^K \rangle \to \langle f^K \phi^* \rangle$ in Eqs. (5) and (8), respectively, where ϕ is the basis function on the Fermi surface with $\langle |\phi|^2 \rangle = 1$. It is then straightforward to show that Eq. (15) still holds, and Eq. (2) with the slow-variation approximation is modified into

$$\mathbf{E} = \mathbf{B} \times \underline{R}_H \mathbf{j}_s. \tag{17}$$

The corresponding Hall coefficient is now a tensor:

$$\underline{R}_H = \frac{1}{2N(0)ec} \left\langle \frac{\partial}{\partial \mathbf{p}_F}(1-Y)\mathbf{v}_F \right\rangle \langle (1-Y)\mathbf{v}_F\mathbf{v}_F \rangle^{-1}, \tag{18}$$

where $Y \equiv Y(\mathbf{p}_F, T)$ denotes the Yosida function[39,40] given in terms of $\varepsilon_n \equiv (2n+1)\pi k_B T$ by

$$Y(\mathbf{p}_F, T) \equiv 1 - 2\pi k_B T \sum_{n=0}^{\infty} \frac{|\Delta|^2 |\phi(\mathbf{p}_F)|^2}{[\varepsilon_n^2 + |\Delta|^2 |\phi(\mathbf{p}_F)|^2]^{3/2}}. \tag{19}$$

The factor $1-Y$ in Eq. (18) acquires angular dependence for the anisotropic pairing at finite temperatures due to the anisotropic distribution of thermally excited quasiparticles embodied in Y.

We realize from Eq. (18) with $Y(\mathbf{p}_F, 0) = 0$ that the superconducting Hall coefficient $\underline{R}_H$ at zero temperature should have the same sign and magnitude for an arbitrary pairing as that of the normal state. It agrees with the expression obtained by Adkins and Waldram at $T = 0$.[10] It is determined essentially by the integration of the curvature of the Fermi energy $\epsilon_F \equiv \epsilon(\mathbf{p}_F)$ over the entire Fermi surface. Especially, $\underline{R}_H$ has no temperature dependence for the s-wave pairing where $1-Y$ in Eq. (18) cancels. In contrast, the gap anisotropy can bring a considerable temperature dependence in $\underline{R}_H$, as may be realized from $1-Y \propto |\phi|^2$ for $T \lesssim T_c$. It is not $\mathbf{v}_F$ itself for $T \lesssim T_c$ but $\mathbf{v}_F |\phi|^2$ that is to be differentiated with respect to $\mathbf{p}_F$. In other words, the anisotropic distribution of thermally excited quasiparticles also plays a crucial role for the superconducting Hall coefficient at finite temperatures. This will be demonstrated in Sec. IV by a model calculation on a d-wave pairing.

We now consider several extensions. When there are internal degrees of freedom in the relevant pairing,[41] we need to change $\Delta(\mathbf{r}) \to \Sigma_i \Delta_i(\mathbf{r})\phi_i(\mathbf{p}_F)$ in Eq. (5) with $\langle \phi_i^* \phi_j \rangle = \delta_{ij}$ as well as $\Delta \to \Delta_i$ and $\langle f^K \rangle \to \langle f^K \phi_i^* \rangle$ in Eq. (8). It can be seen easily that Eqs. (15) and (18) still hold with a modification of $|\Delta|^2 |\phi|^2 \to |\Sigma_i \Delta_i \phi_i|^2$ in Eq. (19). The odd-parity case with $(\uparrow\downarrow, \downarrow\uparrow)$ pairing can be handled similarly with the modifications $\bar{f} \to -\bar{f}$ and $\phi_i^* \to -\bar{\phi}_i$ in the whole formulation. This latter pairing was studied in terms of the superconductivity in Sr_2RuO_4 with a phenomenological Ginzburg-Landau functional to predict a spontaneous Hall effect for a chiral p-wave state.[42]

We next consider the effects of impurities on the s-wave pairing within the Born approximation for the s-wave scattering.[4] This is carried out by adding terms $(i\hbar/2\tau)[\langle \hat{g}^R \rangle, \hat{g}^R]$ and $(i\hbar/2\tau)(\langle \hat{g}^R \rangle \hat{g}^K + \langle \hat{g}^K \rangle \hat{g}^A - \hat{g}^R \langle \hat{g}^K \rangle - \hat{g}^K \langle \hat{g}^A \rangle)$ on the left-hand side of Eq. (4) for g^R and g^K, respectively, with τ denoting the relaxation time. Then one can show that g_1^R still obeys Eq. (14) with $g_{0,1}^K \to g_{0,1}^R$. On the other hand, the equation for g_1^K becomes more complicated to prevent a straightforward extension of the clean-limit consideration. Restricting ourselves to the Ginzburg-Landau region near T_c and carrying out the expansion with respect to Δ,[3] however, one can show that: (i) $g_1^K \sim O(|\Delta|^2)$ whereas $f_1^K \sim O(|\Delta|^3)$; and (ii) g_1^K satisfies Eq. (14). Thus, Eq. (2) is valid near T_c even in the presence of impurities, and also expected to hold approximately at lower temperatures.

We finally comment on the present results in terms of preceding theoretical treatments. A transverse electric field is shown here to result naturally due to the magnetic Lorentz force, in contrast to a treatment based on phenomenologically extended Ginzburg-Landau equations.[43] Compared with those by Hong[11] and Omel'yanchuk and Beloborod'ko[12] for the free-electron model, the present mechanism due to the Lorentz force requires no energy dependence in the density of states near the Fermi level, thereby establishing the general existence of the transverse electric field among superconductors. As for the additional contribution found by Hong[11] and Omel'yanchuk and Beloborod'ko,[12] it is due to the energy dependence in the density of states, accompanied by a reduction in the pair potential, and predicted to vanish at $T = 0$. Hence it may be distinguished clearly from Eq. (17) by experiments on clean type-II superconductors in the Meissner state. There is yet another mechanism of a finite electric field in superconductors not directly connected with the supercurrent, i.e., that caused by a reduction in the pair potential such as the one in a vortex core of type-II superconductors.[33] However, this effect can also be neglected for clean type-II superconductors in the Meissner state.

IV. NUMERICAL EXAMPLES FOR THE HALL COEFFICIENT

To see the importance of the gap anisotropy on the equilibrium Hall coefficient, we here present a model calculation of Eq. (18) for a d-wave pairing. We specifically consider the dimensionless single-particle energy on a two-dimensional square lattice,

$$\epsilon_{\mathbf{p}} = -2(\cos p_x + \cos p_y) + 4t_1(\cos p_x \cos p_y - 1) + 2t_2(\cos 2p_x + \cos 2p_y - 2). \tag{20}$$

with $t_1 = 1/6$ and $t_2 = -1/5$, which forms a band of $-4 \le \epsilon_{\mathbf{p}} \le 4$. This model has been adopted by Kontani and co-workers[44,45] to describe the Fermi surface of cuprate superconductors in theoretically investigating their normal-state Hall coefficients. The Fermi surfaces for the average electron fillings $n = 0.9$, 1.95 per site are shown in Fig. 1. Each of them is given in the extended zone scheme by a singly connected contour around $(p_x, p_y) = (\pi, \pi)$.

The normal-state Hall coefficient $R_H^{(n)} \equiv R_{Hx}^{(n)} = R_{Hy}^{(n)}$ for this band is obtained by estimating Eq. (18) with $Y = 0$. It is found that $R_H^{(n)}$ changes its sign at the filling $n_c = 1.033$ ($\epsilon_F = -0.121$) from negative to positive, as shown in Fig. 2. Thus, the Fermi surface for $n \sim n_c$ consists of competing portions with positive and negative curvatures which almost cancel with each other. It is hence expected that the extra
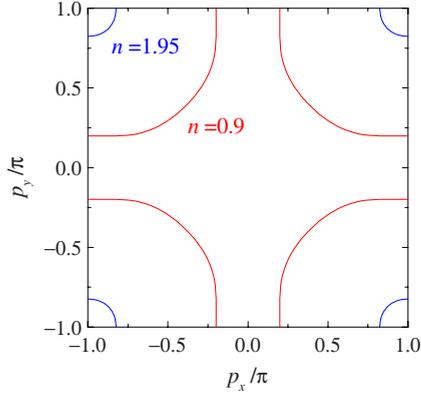
FIG. 1. (Color online) Fermi surfaces of $n=0.9$, 1.95 for the single-particle energy of Eq. (20).

modulation of the curvature at finite temperatures due to the gap anisotropy, embodied in the factor $1-Y$ of Eq. (18), produces the most spectacular effects around $n \sim n_c$.

It should be noted that the Fermi surface by Eq. (20) is not sufficient to account for the signs and temperature dependences of the normal-state Hall coefficient in high-T_c superconductors, especially the positive sign of $R_H^{(n)}$ observed in $Nd_{2-x}Ce_xCuO_4$.[46] Indeed, the vertex corrections due to the strong antiferromagnetic fluctuations have been shown crucial for explaining the observed behaviors of $n \sim 1$.[44,45] However, we expect that the single-particle model adopted here will be sufficient to capture the essential physics which the gap anisotropy brings into the superconducting Hall coefficient.

To see this, we here adopt a model d-wave pairing appropriate for $n \gtrsim 0.8$,

$$\phi(\mathbf{p}_F) = A[(p_{Fx} - \pi)^2 - (p_{Fy} - \pi)^2], \qquad (21)$$

where A is the normalization constant determined by $\langle |\phi|^2 \rangle = 1$.

Figure 3 displays $R_H \equiv R_{Hx} = R_{Hy}$ of the equilibrium supercurrent for $n=0.9$, 1.95 calculated by Eq. (18). It is normalized by the normal-state Hall coefficient $R_H^{(n)}$ for convenience. With $n=1.95$ where the Fermi surface is almost isotropic and free-electron-like with positive charge, R_H increases monotonically from $R_H/R_H^{(n)}=1.0$ at $T=0$ to $R_H/R_H^{(n)} \sim 3.0$ at $T=T_c$. On the other hand, $R_H/R_H^{(n)}$ for n
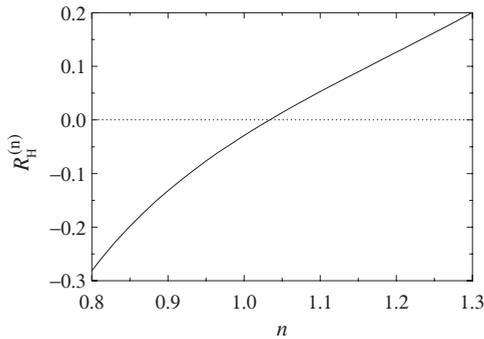
FIG. 2. The normal-state Hall coefficient $R_H^{(n)}$ as a function of the filling n for the single-particle dispersion of Eq. (20).
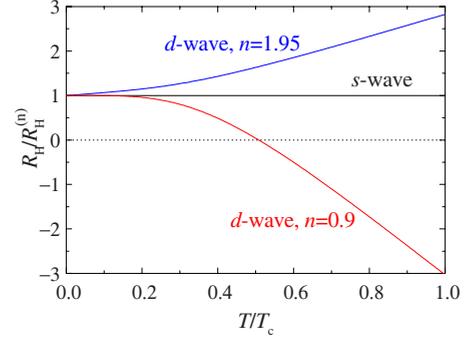
FIG. 3. (Color online) The Hall coefficient R_H of equilibrium supercurrents, normalized by the normal-state coefficient $R_H^{(n)}$, as a function of temperature for the s-wave pairing, the d-wave pairing with $n=0.9$, and the d-wave pairing with $n=1.95$.

$=0.9$ even changes the sign as the temperature is increased from $T=0$. These strong temperature dependences are brought about by the modulation of the Fermi-surface curvature by the anisotropic distribution of thermally excited quasiparticles.

The sign change in the Hall coefficient as a function of temperature/magnetic field has been observed in several high-T_c superconductors in the vortex state with dissipative currents.[47–53] The origin of this sign change still remains mysterious because any attempt to analyze it theoretically necessarily has to clarify complicated vortex motions of type-II superconductors with electromagnetic fields. On the other hand, we have shown here that the sign change can occur even in the Hall coefficient of equilibrium supercurrents, which is much simpler without vortex motions, due to the modification of the Fermi-surface curvature at finite temperatures caused by the anisotropic distribution of thermally excited quasiparticles. It may be detected in some high-T_c superconductors. An observation of this sign change will provide: (i) an unambiguous support for the mechanism clarified here; and (ii) a clue to understand the sign change in the vortex state with dissipative currents. We note in this context that neither the magnetic Lorentz force nor the gap anisotropy were incorporated in the phenomenological theories on the vortex motion[21,29,33,50,51] and in the microscopic theory based on the time-dependent Ginzburg-Landau equations.[32]

V. SUMMARY

We have developed a microscopic theory of the Lorentz force in equilibrium superconductors using a theoretical framework which embraces the normal-state Boltzmann equation. The magnetic Lorentz force working on equilibrium supercurrents is shown to induce an electric field as Eq. (17), which has the same expression as the normal-state one with dissipative currents. Using the slow-variation approximation appropriate for type-II superconductors, we have obtained an analytic expression for the Hall coefficient in the clean limit as Eq. (18). It tells us that: (i) R_H at $T=0$ carries the same sign and magnitude as that of the normal state; (ii) the coefficient stays constant up to T_c for the s-wave pairing;

and (iii) R_H can have a considerable temperature dependence for a nonisotropic energy gap due to the anisotropic quasiparticle distribution at finite temperatures. We have shown in terms of the point (iii) that a sign change in R_H may result, as seen in Fig. 3. This sign change in the equilibrium Hall coefficient may be observed in some high-T_c superconductors in the Meissner state. The present mechanism for the sign change, which has not been considered in any of the preceding treatments, may also play an essential role in the sign change of R_H observed in the vortex state with dissipative currents.[47–53]

Further experiments for a wide range of materials are desired on the Hall voltage in the Meissner state for probing the Lorentz force through the sign and magnitude of the superconducting Hall coefficient. The electric field will also be present in the vortex-lattice state to form a long-range periodic pattern, which may in principle be detected by experiment.

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